Congruenze lineari

Leonardo Bizzoni

October 10, 2023

Si dice **congruenza lineare** modulo n ogni espressione della forma:

$$ax \equiv b \bmod n$$

con $a, b \in \mathbb{Z}$, $n \in \mathbb{Z} \geq 1$ e $x \in \mathbb{Z}$ incognita.

Si dice soluzione ogni intero c tale che verifica l'espressione data.

1 Esercizi

Siano $a, b, c, d, n \in \mathbb{Z}$ con $n \geq 1$ e

$$a \equiv b \bmod n$$
$$c \equiv d \bmod n$$

Dimostrare che $a + c \equiv b + d \bmod n$

$$ac \equiv bd \bmod n$$

$$\left.\begin{array}{l} a \equiv b \bmod n \Rightarrow n \mid a-b \Rightarrow a-b = nk_1 \\ c \equiv d \bmod n \Rightarrow n \mid c-d \Rightarrow c-d = nk_2 \end{array}\right. \quad \text{con } k_1, k_2 \in \mathbb{Z}$$

$$a+c \equiv b+d \bmod n \Rightarrow n \mid a+c-b-d \Rightarrow n \mid (a-b)+(c-d) \Rightarrow n \mid nk_1 + nk_2 \Rightarrow n \mid n(k_1+k_2)$$

$$ac \equiv bd \bmod n \Rightarrow n \mid ac - bd \quad ???$$

$$35x \equiv 23 \bmod 16 \Rightarrow 35x + 16y = 23$$

$(35, 16) = 1 \qquad 1 \mid 23 \checkmark$

$35 = 16 \cdot 2 + 3$	$3 = a - 2b$
$16 = 3 \cdot 5 + 1$	$1 = b - 5(a - 2b) = 11b - 5a$
$3 = 1 \cdot 3 + 0$	$23 = 23(11b) + 23(-5a)$
	$= 16(253) + 35(-115)$

Tutte le soluzioni avranno questa forma

$c = -115$

$$-115 + \frac{16}{1}k = -115 + 16k \quad \text{con } k \in \mathbb{Z}$$

$$75x \equiv 35 \bmod 29$$

$$75x + 29y = 35$$

$(75, 29) = 1 \qquad 1 \mid 35 \checkmark$

$29 \cdot 2 = 58$

$75 = 29 \cdot 2 + 17$ ($75 - 58 = 17$)	$17 = a - 2b$
$29 = 17 \cdot 1 + 12$ ($29 - 17 = 12$)	$12 = 3b - a$

$14 = 12 \cdot 1 + 5$ $\quad$ $5 = 2a - 3b$

$12 = 5 \cdot 2 + 2$ $\quad$ $2 = 13b - 5a$

$5 = 2 \cdot 2 + 1$ $\quad$ $1 = 12a - 31b$

$2 = 1 \cdot 2 + 0$ $\quad$ $35 = 420a - 1085b$

$35 \cdot 12 = 70, 350$ $\quad$ $35 \cdot 31 = 35, 1050$

$c = 420$

$420 + \frac{29}{1}K = 420 + 29K, K \in \mathbb{Z}$

$148x \equiv 14 \mod 352$

$148x + 352y = 14$

$(352, 148) = 4$ $\quad$ $4 \nmid 14$ quindi la congruenza lineare non ha soluzioni

$148 \cdot 2 = 296$ $\quad$ $352 - 296 = 56$

$352 = 148 \cdot 2 + 56$

$56 \cdot 2 = 112$ $\quad$ $148 - 112 = 36$

$148 = 56 \cdot 2 + 36$

$56 = 36 \cdot 1 + 20$

$36 = 20 \cdot 1 + 16$

$20 = 16 \cdot 1 + 4$

$16 = 4 \cdot 4 + 0$

$72x \equiv 48 \mod 200$

$72x + 200y = 48$

$(200, 72) = 8$ $\quad$ $8 \mid 48$

$200 = 72 \cdot 2 + 56$ $\quad$ ($200 - 144 = 56$) $\quad$ $56 = a - 2b$

$72 = 56 \cdot 1 + 16$ $\quad$ ($72 - 56 = 16$) $\quad$ $16 = 3b - a$

$16 \cdot 3 = 48$

$56 = 16 \cdot 3 + 8$ $\quad$ ($56 - 48 = 8$) $\quad$ $8 = a - 2b - 3(3b - a) = 4a - 11b$

$16 = 8 \cdot 2 + 0$ $\quad$ $48 = 6(4a - 11b) = 24a - 66b$

$c = -66$

$-66 + \frac{200}{8}K = -66 + 25K, K \in \mathbb{Z}$

$200 : 8 = 25$ (resto 40)

$144x \equiv 81 \mod 387$

$144x + 387y = 81$ $\qquad$ $9 \mid 81$ $\qquad$ $81 \mid 9$, $0 \mid 9$

$(387, 144) = 9$

$144 \cdot 3 = 432$

$387 = 144 \cdot 2 + 99$ $\qquad$ $387 - 288 = 99$ $\qquad$ $99 = a - 2b$

$144 = 99 \cdot 1 + 45$ $\qquad$ $144 - 99 = 45$ $\qquad$ $45 = 3b - a$

$99 = 45 \cdot 2 + 9$ $\qquad$ $9 = 3a - 8b$

$45 = 9 \cdot 5 + 0$ $\qquad$ $81 = 9(3a - 8b) = 27a - 72b$

$x = -72$

$-72 + \frac{387}{9}K = -72 + 43K$

$246x \equiv 66 \mod 702$

$246x + 702y = 66$

$(702, 246) = 6$ $\qquad$ $6 \mid 66$

$246 \cdot 2 = 492$ $\qquad$ $702 - 492 = 210$

$702 = 246 \cdot 2 + 210$ $\qquad$ $210 = a - 2b$

$246 = 210 \cdot 1 + 36$ $\qquad$ $36 = b - a + 2b = 3b - a$

$36 \cdot 5 = 180$

$210 = 36 \cdot 5 + 30$ $\qquad$ $30 = a - 2b - 5(3b - a) = 6a - 17b$

$36 = 30 + 6$ $\qquad$ $6 = 20b - 7a$

$30 = 6 \cdot 5 + 0$ $\qquad$ $66 = 220b - 77a$

$x = 220$

$220 + \frac{702}{6}K = 220 + 117K, \ K \in \mathbb{Z}$

$702 \mid 6$, 102, 42, 0 $\qquad$ 117

$27x \equiv 20 \mod 751$

$27x + 151y = 20$

$(151, 27) = 1 \qquad 1 \mid 20$

$$\begin{array}{r} 27\cdot \\ 5 \\ \hline 135 \end{array}$$

$151 = 27 \cdot 5 + 16$ $\quad \begin{array}{r} 151- \\ 135 \\ \hline 16 \end{array}$ $\qquad 16 = a - 5b$

$27 = 16 \cdot 1 + 11$ $\quad \begin{array}{r} 27- \\ 16 \\ \hline 11 \end{array}$ $\qquad 11 = 6b - a$

$16 = 11 + 5 \qquad 5 = 2a - 11b$

$11 = 5 \cdot 2 + 1 \qquad 1 = 6b - a - 2(2a - 11b) = 28b - 5a$

$5 = 1 \cdot 5 + 0 \qquad 20 = 560b - 100a$

$$\begin{array}{r} 28\cdot \\ 20 \\ \hline 00 \\ 560 \end{array}$$

$x = 560$

$560 + \frac{151}{1}k = 560 + 151k, \; k \in \mathbb{Z}$
